Unit 4: Full Factorial Experiments at Two Levels

Source : Chapter 4 (sections 4.1-4.4, 4.6-4.12, 4.15).

- An Epitaxial Layer Growth Experiment (Section 4.1).
- Basic concepts for 2^k designs (Section 4.2).
- Factorial effects and plots (Section 4.3).
- Using Regression to Compute Factorial Effects (Section 4.4).
- Fundamental principles (Section 4.6).
- Comparisons with ”one-factor-at-a-time” approach (Section 4.7).
- Normal and Half-normal plots for detecting effect significance (Section 4.8).
- Lenth’s Method (Section 4.9).
- Nominal-the best problem, quadratic loss function (Section 4.10).
- Use of Log Sample Variance for Dispersion Analysis (Section 4.11).
- Analysis of Location and Dispersion: Epitaxial Growth Experiment (Section 4.12).
- Blocking in 2^k design (Section 4.15).

Epitaxial Layer Growth Experiment

- An AT&T experiment based on 2^4 design; four factors each at two levels. There are 6 replicates for each of the 16 ($=2^4$) level combinations; data given on the next page.

Table 1: Factors and Levels, Adapted Epitaxial Layer Growth Experiment

Factor		Level $-$	$+$
A.	susceptor-rotation method	continuous	oscillating
B.	nozzle position	2	6
C.	deposition temperature (°C)	1210	1220
D.	deposition time	low	high

- **Objective :** Reduce variation of y (=layer thickness) around its target 14.5 μm by changing factor level combinations.

Data from Epitaxial Layer Growth Experiment

Table 2: Design Matrix and Thickness Data, Adapted Epitaxial Layer Growth Experiment

Run	Factor A	B	C	D	Thickness						$\bar{y}$	s^2	$\ln s^2$
1	−	−	−	+	14.506	14.153	14.134	14.339	14.953	15.455	14.59	0.270	-1.309
2	−	−	−	−	12.886	12.963	13.669	13.869	14.145	14.007	13.59	0.291	-1.234
3	−	−	+	+	13.926	14.052	14.392	14.428	13.568	15.074	14.24	0.268	-1.317
4	−	−	+	−	13.758	13.992	14.808	13.554	14.283	13.904	14.05	0.197	-1.625
5	−	+	−	+	14.629	13.940	14.466	14.538	15.281	15.046	14.65	0.221	-1.510
6	−	+	−	−	14.059	13.989	13.666	14.706	13.863	13.357	13.94	0.205	-1.585
7	−	+	+	+	13.800	13.896	14.887	14.902	14.461	14.454	14.40	0.222	-1.505
8	−	+	+	−	13.707	13.623	14.210	14.042	14.881	14.378	14.14	0.215	-1.537
9	+	−	−	+	15.050	14.361	13.916	14.431	14.968	15.294	14.67	0.269	-1.313
10	+	−	−	−	14.249	13.900	13.065	13.143	13.708	14.255	13.72	0.272	-1.302
11	+	−	+	+	13.327	13.457	14.368	14.405	13.932	13.552	13.84	0.220	-1.514
12	+	−	+	−	13.605	13.190	13.695	14.259	14.428	14.223	13.90	0.229	-1.474
13	+	+	−	+	14.274	13.904	14.317	14.754	15.188	14.923	14.56	0.227	-1.483
14	+	+	−	−	13.775	14.586	14.379	13.775	13.382	13.382	13.88	0.253	-1.374
15	+	+	+	+	13.723	13.914	14.913	14.808	14.469	13.973	14.30	0.250	-1.386
16	+	+	+	−	14.031	14.467	14.675	14.252	13.658	13.578	14.11	0.192	-1.650

2^k Designs: A General discussion

- $2 \times 2 \times \ldots \times 2 = 2^k$ design.
- Planning matrix vs model matrix (see Tables 4.3, 4.5).
- Run order and restricted randomization (see Table 4.4).
- *Balance*: each factor level appears the same number of times in the design.
- *Orthogonality* : for any pair of factors, each possible level combination appears the same number of times in the design.
- Replicated vs unreplicated experiment.

Main effects and Plots

- Main effect of factor A:

$$\mathrm{ME}(A) = \bar{z}(A+) - \bar{z}(A-).$$

- Advantages of factorial designs (R.A.Fisher): *reproducibility* and *wider inductive basis* for inference.
- Informal analysis using the **main effects plot** can be powerful.

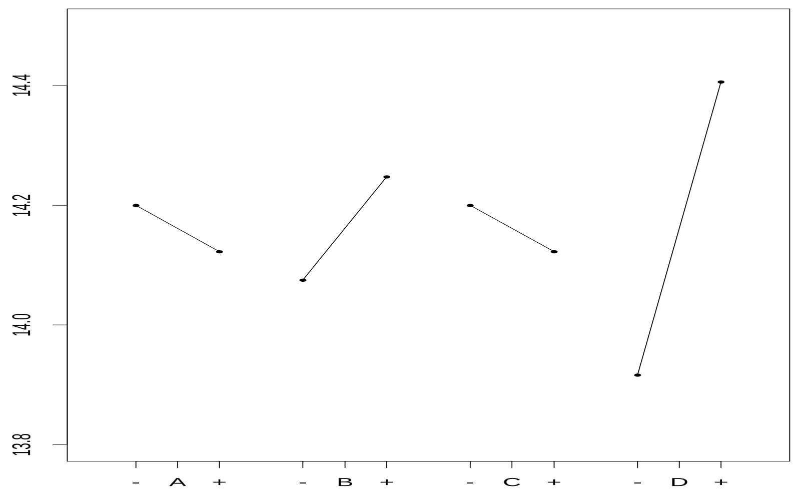

Figure 1: Main Effects Plot, Adapted Epitaxial Layer Growth Experiment

Interaction Effects

- **Conditional main effect** of B at $+$ level of A:

$$\mathrm{ME}(B|A+) = \bar{z}(B+|A+) - \bar{z}(B-|A+).$$

- **Two-factor interaction effect** between A and B:

$$\begin{aligned} INT(A,B) &= \frac{1}{2}\{\mathrm{ME}(B|A+) - \mathrm{ME}(B|A-)\} \\ &= \frac{1}{2}\{\mathrm{ME}(A|B+) - \mathrm{ME}(A|B-)\} \\ &= \frac{1}{2}\{\bar{z}(A+|B+) + \bar{z}(A-|B-)\} - \frac{1}{2}\{\bar{z}(A+|B-) + \bar{z}(A-|B+)\}, \end{aligned} \tag{1}$$

The first two definitions in (1) give more insight on the term ”interaction” than the third one in (1). The latter is commonly used in standard texts.

Interaction Effect Plots

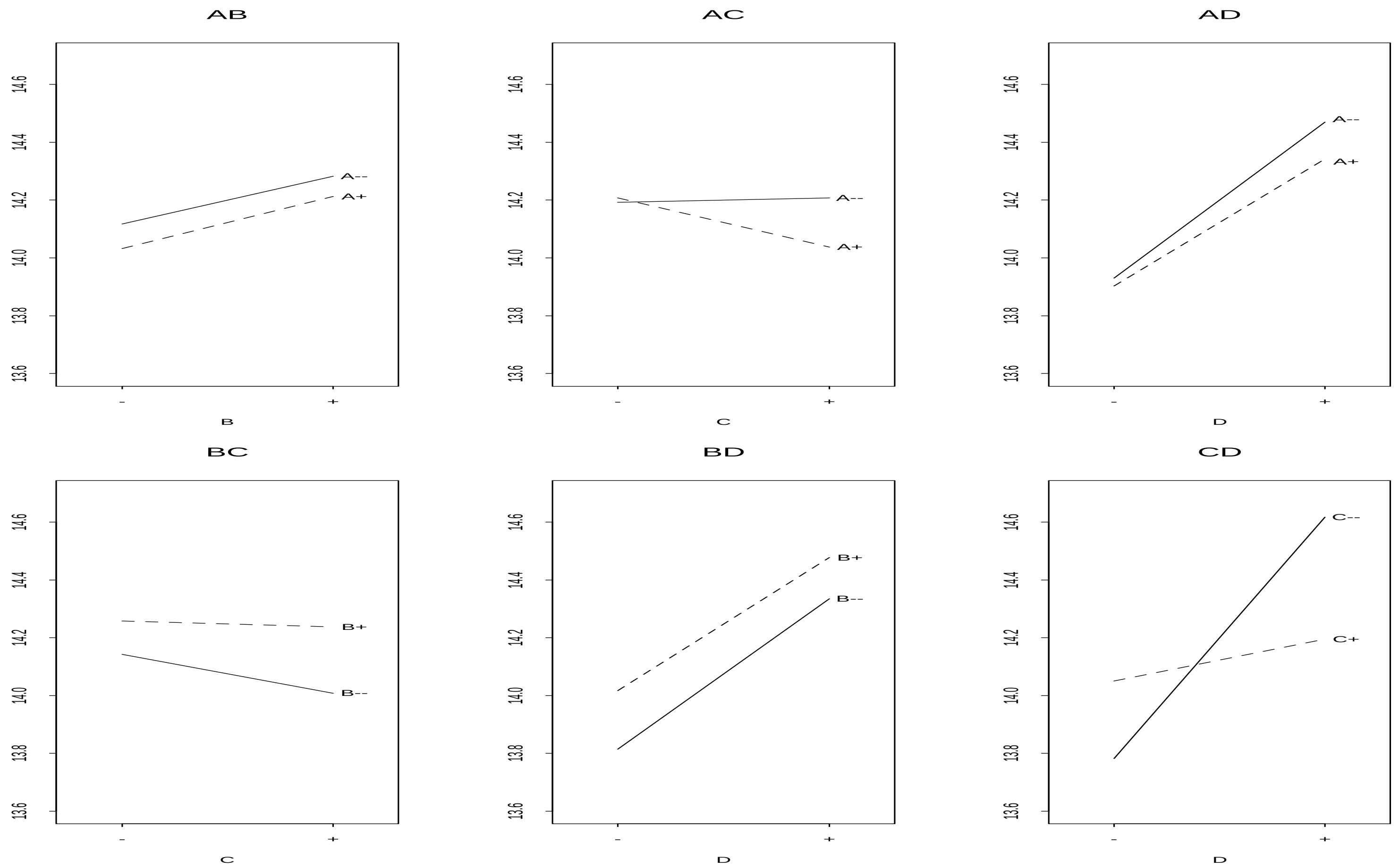

Figure 2: Interaction Plots, Adapted Epitaxial Layer Growth Experiment

Synergistic and Antagonistic Plots

- An A-against-B plot is *synergystic* if $\mathrm{ME}(B|A+)\mathrm{ME}(B|A-) > 0$ and *antagonistic* if $\mathrm{ME}(B|A+)\mathrm{ME}(B|A-) < 0$.

  An antagonistic plot suggests a more *complex* underlying relationship than what the data reveal.

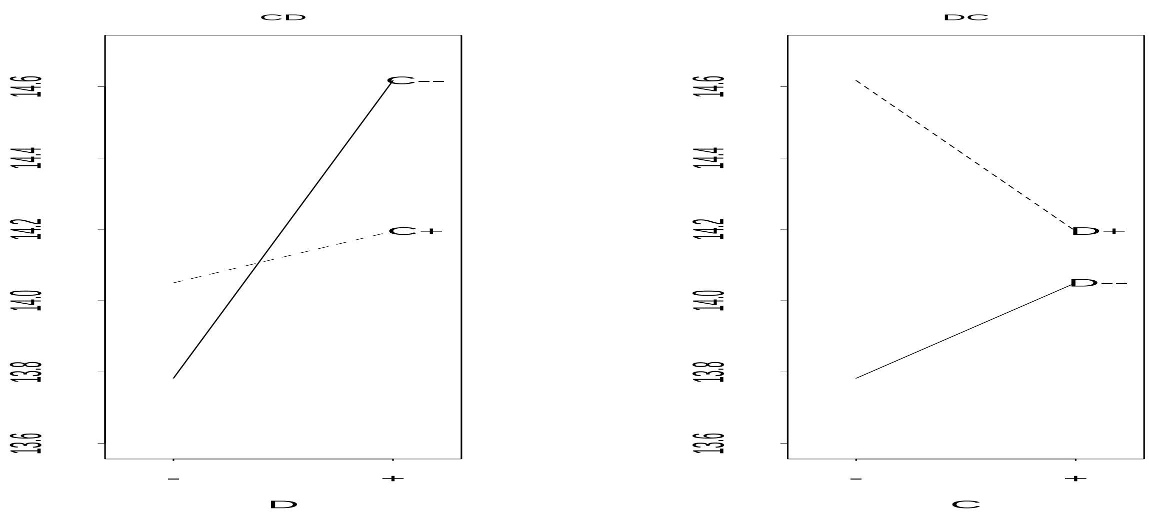

Figure 3: C-against-D and D-against-C Plots, Adapted Epitaxial Layer Growth Experiment

More on Factorial Effects

$$\begin{aligned}\mathrm{INT}(A,B,C) &= \frac{1}{2}\mathrm{INT}(A,B|C+) - \frac{1}{2}\mathrm{INT}(A,B|C-) = \frac{1}{2}\mathrm{INT}(A,C|B+) \\ &\quad -\frac{1}{2}\mathrm{INT}(A,C|B-) = \frac{1}{2}\mathrm{INT}(B,C|A+) - \frac{1}{2}\mathrm{INT}(B,C|A-).\end{aligned}$$

$$\mathrm{INT}(A_1,A_2,\ldots,A_k) = \frac{1}{2}\mathrm{INT}(A_1,A_2,\ldots,A_{k-1}|A_k+) - \frac{1}{2}\mathrm{INT}(A_1,A_2,\ldots,A_{k-1}|A_k-).$$

- A general factorial effect

$$\hat{\theta} = \bar{z}_+ - \bar{z}_-,$$

where $\bar{z}_+$ and $\bar{z}_-$ are averages of one half of the observations. If N is the total number of observations,

$$Var(\hat{\theta}) = \frac{\sigma^2}{N/2} + \frac{\sigma^2}{N/2} = \frac{4}{N}\sigma^2,$$

σ^2 = variance of an observation.

Using Regression Analysis to Compute Factorial Effects

Consider the 2^3 design for factors A, B and C, whose columns are denoted by $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{x_3}$ (=1 or -1).

The interactions AB, AC, BC, ABC are then equal to

$$\mathbf{x_4} = \mathbf{x_1}\mathbf{x_2}, \mathbf{x_5} = \mathbf{x_1}\mathbf{x_3}, \mathbf{x_6} = \mathbf{x_2}\mathbf{x_3}, \mathbf{x_7} = \mathbf{x_1}\mathbf{x_2}\mathbf{x_3} \text{ (see Table 3).}$$

Use the regression model

$$z_i = \beta_0 + \sum_{j=1}^{7} \beta_j x_{ij} + \varepsilon_i,$$

where $i = i^{th}$ observation.

The regression (i.e., least squares) estimate of β_j is

$$\hat{\beta}_j = \frac{1}{1-(-1)}\left(\bar{z}(x_{ij}=+1) - \bar{z}(x_{ij}=-1)\right)$$

$$= \tfrac{1}{2}(\text{factorial effect of variable } x_j)$$

Model Matrix for 2^3 Design

Table 3: Model Matrix for 2^3 Design

1 A	2 B	3 C	12 AB	13 AC	23 BC	123 ABC
$-$	$-$	$-$	$+$	$+$	$+$	$-$
$-$	$-$	$+$	$+$	$-$	$-$	$+$
$-$	$+$	$-$	$-$	$+$	$-$	$+$
$-$	$+$	$+$	$-$	$-$	$+$	$-$
$+$	$-$	$-$	$-$	$-$	$+$	$+$
$+$	$-$	$+$	$-$	$+$	$-$	$-$
$+$	$+$	$-$	$+$	$-$	$-$	$-$
$+$	$+$	$+$	$+$	$+$	$+$	$+$

Factorial Effects, Adapted Epi-Layer Growth Experiment

Table 4: Factorial Effects, Adapted Epitaxial Layer Growth Experiment

Effect	$\bar{y}$	$\ln s^2$
A	-0.078	0.016
B	0.173	-0.118
C	-0.078	-0.112
D	0.490	0.056
AB	0.008	0.045
AC	-0.093	-0.026
AD	-0.050	-0.029
BC	0.058	0.080
BD	-0.030	0.010
CD	-0.345	0.085
ABC	0.098	-0.032
ABD	0.025	0.042
ACD	-0.030	0.000
BCD	0.110	-0.003
$ABCD$	0.020	0.103

Fundamental Principles in Factorial Design

- **Effect Hierarchy Principle**

  (i) Lower order effects are more likely to be important than higher order effects.

  (ii) Effects of the same order are equally likely to be important.

- **Effect Sparsity principle** (Box-Meyer)

  The number of relatively important effects in a factorial experiment is small.

  This is similar to the *Pareto Principle* in quality investigation.

  Effect hierarchy and sparsity principles are more effective/relevant for *screening experiments*.

- **Effect Heredity Principle** (Hamada-Wu)

  In order for an interaction to be significant, at least one of its parent factors should be significant.

  For modeling, McCullagh and Nelder called it the *Marginality Principle*.

One-Factor-At-A-Time (ofat) Approach

Table 5: Planning Matrix for 2^3 Design and Response Data For Comparison with One-Factor-At-A-Time Approach

Factor			Percent
P	R	S	Burned
1200	0.3	slow	11
1200	0.3	fast	17
1200	0.6	slow	25
1200	0.6	fast	29
1400	0.3	slow	02
1400	0.3	fast	09
1400	0.6	slow	37
1400	0.6	fast	40

One-Factor-At-A-Time (ofat) Approach (Contd.)

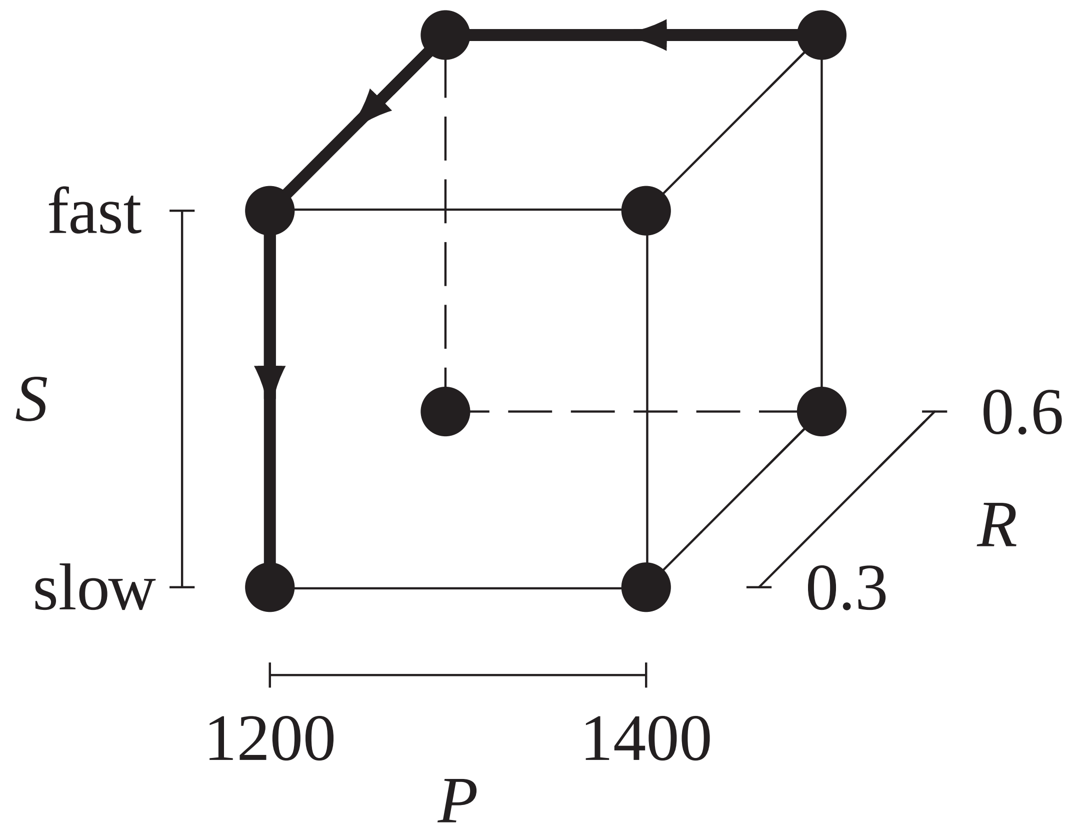

Figure 4: The Path of a One-Factor-At-A-Time Plan

The three steps of ofat as illustrated in the arrows in Figure 4 are detailed in steps 1-3 on page 174 of WH.

Disadvantages of ofat Approach Relative to Factorial Approach

1. *It requires more runs for the same precision in effect estimation.* In the example, the 2^3 design requires 8 runs. For ofat to have the same precision, each of the 4 corners on the ofat path needs to have 4 runs, totaling 16 runs. In general, to be comparable to a 2^k design, ofat would require 2^{k-1} runs at each of the $k+1$ corners on its path, totaling $(k+1)2^{k-1}$. The ratio is $(k+1)2^{k-1}/2^k = (k+1)/2$.

2. *It cannot estimate some interactions*.

3. *Conclusions for analysis not as general*.

4. It can miss optimal settings.

For points $2-4$, see Figure 4.

Why Experimenters Continue to Use ofat?

- Most physical laws are taught by varying one factor at a time. Easier to think and focus on one factor each time.
- Experimenters often have good intuition about the problem when thinking in this mode.
- No exposure to statistical design of experiments.
- **Challenges** for DOE researchers: To combine the factorial approach with the good intuition rendered by the the ofat approach. Needs a new outlook.

Normal Plot of Factorial Effects

- Suppose $\hat{\theta}_i, i = 1, \cdots, I$, are the factorial effect estimates (example in Table 4). Order them as $\hat{\theta}_{(1)} \leq \cdots \leq \hat{\theta}_{(I)}$. **Normal probability plot** (see Unit 2):

$$\hat{\theta}_i \text{ (vertical) vs. } \Phi^{-1}([i-0.5]/I) \text{ (horizontal)}$$

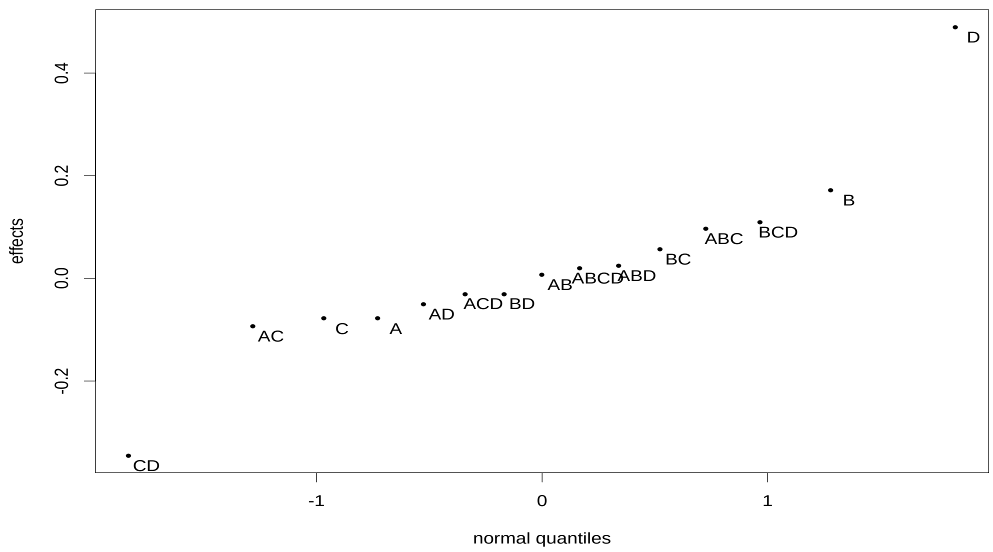

Figure 5: Normal Plot of Location Effects, Adapted Epitaxial Layer Growth Experiment

Use of Normal Plot to Detect Effect Significance

- **Deduction Step.** Null hypothesis H_0 : all factorial effects = 0 . Under H_0, $\hat{\theta}_i \sim N(0, \sigma^2)$ and the resulting normal plot should follow a *straight line*.

- **Induction Step.** By fitting a straight line to the middle group of points (around 0) in the normal plot, *any effect whose corresponding point falls off the line is declared significant* (Daniel, 1959).

- Unlike t or F test, no estimate of σ^2 is required. Method is especially suitable for *unreplicated* experiments. In t test, s^2 is the *reference quantity*. For unreplicated experiments, Daniel's idea is to use the *normal curve as the reference distribution*.

- In Figure 5, D, CD (and possibly B?) are significant. Method is informal and judgemental.

Normal and Half Normal Plots

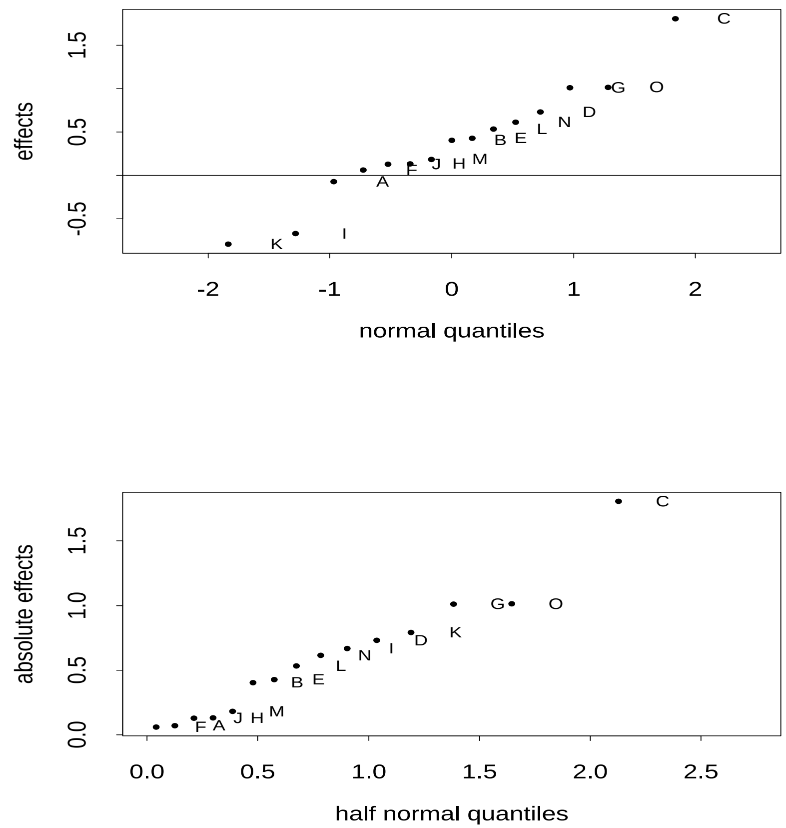

Figure 6: Comparison of Normal and Half-Normal Plots

Visual Misjudgement with Normal Plot

Potential misuse of normal plot :

In Figure 6 (top), by following the procedure for detecting effect significance, one may declare C, K and I are significant, because they "deviate" from the middle straight line. This is *wrong* because it ignores the obvious fact that K and I are smaller than G and O in magnitude. This points to a potential visual misjudgement and misuse with the normal plot.

Half-Normal Plot

- **Idea:** Order the absolute $\hat{\theta}_{(i)}$ values as $|\hat{\theta}|_{(1)} \leq \cdots |\hat{\theta}|_{(I)}$ and plot them on the positive axis of the normal distribution (thus the term "half-normal"). This would avoid the potential misjudgement between the positive and negative values.
- The **half-normal probability plot** consists of the points

$$(\Phi^{-1}(0.5+0.5[i-0.5]/I), |\hat{\theta}|_{(i)}), \text{ for } i = 1, \ldots, 2^k - 1. \tag{2}$$

- In Figure 6 (bottom), only C is declared significant. Notice that K and I no longer stand out in terms of the absolute values.
- For the rest of the book, **half-normal plots will be used for detecting effect significance**

A Formal Test of Effect Significance : Lenth's Method

- Sometimes it is desirable to have a formal test that can assign p values to the effects. The following method is also available in packages like SAS.
- Lenth's Method

  1. Compute the pseudo standard error

     $$PSE = 1.5 \cdot \text{median}_{\{|\hat{\theta}_i| < 2.5 s_0\}} |\hat{\theta}_i|,$$

     where the median is computed among the $|\hat{\theta}_i|$ with $|\hat{\theta}_i| < 2.5 s_0$ and

     $$s_0 = 1.5 \cdot \text{median}|\hat{\theta}_i|.$$

     (Justification : If $\theta_i = 0$ and error is normal, s_0 is a *consistent* estimate of the standard deviation of $\hat{\theta}_i$. Use of median gives "robustness" to outlying values.)

A Formal Test of Effect Significance (Contd.)

2. Compute

$$t_{PSE,i} = \frac{\hat{\theta}_i}{PSE}, \quad \text{for each } i.$$

If $|t_{PSE,i}|$ exceeds the critical value given in Appendix H (or from software), $\hat{\theta}_i$ is declared significant.

- Two versions of the critical values are considered next.

Two Versions of Lenth's Method

- **Individual Error Rate (IER)**

  H_0 : all θ_i's $= 0$, normal error.

  IER_α at level α is determined by

$$Prob(|t_{PSE,i}| > \text{IER}_\alpha | H_0) \quad = \quad \alpha, \quad \text{for } i = 1, \cdots, I.$$

  (Note : Because $\theta_i = 0$, $t_{PSE,i}$ has the *same* distribution under H_0 for all i.)

- **Experiment-wise Error Rate (EER)**

$$\begin{aligned} &Prob(|t_{PSE,i}| > \text{EER}_\alpha \ \text{ for at least one } i, i = 1, \ldots, I | H_0) \\ &\qquad = Prob(\max_{1 \le i \le I} |t_{PSE,i}| > \text{EER}_\alpha | H_0) = \alpha. \end{aligned}$$

- EER accounts for the number of tests done in the experiment but often gives conservative results (less powerful). In screening experiments, IER is more powerful and preferable because many of the θ_i's are negligible (recall the effect sparsity principle). The EER critical values can be inflated by considering many θ_i values. (Why?)

Illustration with Adapted Epi-Layer Growth Experiment

1. In Table 4, median$|\hat{\theta}_i|$ = 0.078, $s_0 = 1.5 \times 0.078$ = 0.117.
   Trimming constant $2.5s_0 = 2.5 \times 0.117 = 0.292$, which eliminates 0.490 (D) and 0.345 (CD).
   Then $\text{median}_{\{|\hat{\theta}_i|<2.5s_0\}}|\hat{\theta}_i|$ = 0.058, $PSE = 1.5 \times 0.058 = 0.087$.
   The corresponding $|t_{PSE}|$ values appear in Table 6.

2. For $\alpha = 0.01$, $\text{IER}_{0.01} = 3.63$ for $I = 15$. By comparing with the $|t_{PSE}|$ values, D and CD are significant at 0.01 level. Use of $\text{EER}_{0.01} = 6.45$ (for $I = 15$) will not detect any effect significance. Analysis of the $|t_{PSE}|$ values for $\ln s^2$ (Table 6) detects no significant effect (details on page 182 of WH), thus confirming the half-normal plot analysis in Figure 4.10 of section 4.8.

- p values of effects can be obtained from packages or by interpolating the critical values in the tables in appendix H. (See page 182 for illustration).

$|t_{PSE}|$ Values for Adapted Epi-Layer Growth Experiment

Table 6: $|t_{PSE}|$ Values, Adapted Epitaxial Layer Growth Experiment

Effect	$\bar{y}$	$\ln s^2$
A	0.90	0.25
B	1.99	1.87
C	0.90	1.78
D	5.63	0.89
AB	0.09	0.71
AC	1.07	0.41
AD	0.57	0.46
BC	0.67	1.27
BD	0.34	0.16
CD	3.97	1.35
ABC	1.13	0.51
ABD	0.29	0.67
ACD	0.34	0.00
BCD	1.26	0.05
ABCD	0.23	1.63

Nominal-the-Best Problem

- There is a nominal or target value t based on engineering design requirements. Define a quantitative loss due to deviation of y from t.

  Quadratic loss : $L(y,t) = c(y-t)^2$.
  $E(L(y,t)) = cVar(y) + c(E(y)-t)^2$.

- **Two-step procedure for nominal-the-best problem**:
  (i) Select levels of some factors to minimize $Var(y)$.
  (ii)Select the level of a factor not in (i) to move $E(y)$ closer to t.
  A factor in step (ii) is an *adjustment* factor if it has a significant effect on $E(y)$ but not on $Var(y)$. Procedure is effective only if an adjustment factor can be found. This is often done on engineering ground. (Examples of adjustment factors : deposition time in surface film deposition process, mold size in tile fabrication, location and spacing of markings on the dial of a weighing scale).

Why Take $\ln s^2$?

- It maps s^2 over $(0,\infty)$ to $\ln s^2$ over $(-\infty,\infty)$. Regression and ANOVA assume the responses are nearly normal, i.e. over $(-\infty,\infty)$.
- Better for variance prediction. Suppose $z = \ln s^2$. $\hat{z}$ = predicted value of $\ln\sigma^2$, then $e^{\hat{z}}$ = predicted variance of σ^2, always nonnegative.
- Most physical laws have a multiplicative component. Log converts *multiplicity* into *additivity*.
- Variance stabilizing property: next page.

$\ln s^2$ as a Variance Stabilizing Transformation

- Assume $y_{ij} \sim N(0, \sigma_i^2)$. Then $(n_i - 1)s_i^2 = \sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2 \sim \sigma_i^2 \chi^2_{n_i - 1}$,

$$\ln s_i^2 = \ln \sigma_i^2 + \ln(\chi^2_{n_i-1}/(n_i - 1)). \tag{3}$$

- X a random variable, h a smooth function,

$$\mathrm{Var}(h(X)) \approx [h'(E(X))]^2 \mathrm{Var}(X) \tag{4}$$

- Take $X = \frac{\chi^2_\nu}{\nu}$ and $h = \ln$. Then $E(X) = 1$ and $\mathrm{Var}(X) = \frac{2}{\nu}$.
- Applying (3) to $X = \frac{\chi^2_\nu}{\nu}$ leads to

$$\mathrm{Var}(\ln(X)) \approx [h'(1)]^2 \frac{2}{\nu} = \frac{2}{\nu}.$$

In (2), $\nu = n_i - 1$, $\ln s_i^2 \sim N(\ln \sigma_i^2, 2(n_i - 1)^{-1})$. The variance of $\ln s_i^2$, $2(n_i - 1)^{-1}$, is nearly constant for $n_i - 1 \geq 9$.

Epi-layer Growth Experiment Revisited

Original data from Shoemaker, Tsui and Wu (1991).

Table 7: Design Matrix and Thickness Data, Original Epitaxial Layer Growth Experiment

Design				Thickness						$\bar{y}$	s^2	$\ln s^2$
A	B	C	D									
−	−	−	+	14.812	14.774	14.772	14.794	14.860	14.914	14.821	0.003	-5.771
−	−	−	−	13.768	13.778	13.870	13.896	13.932	13.914	13.860	0.005	-5.311
−	−	+	+	14.722	14.736	14.774	14.778	14.682	14.850	14.757	0.003	-5.704
−	−	+	−	13.860	13.876	13.932	13.846	13.896	13.870	13.880	0.001	-6.984
−	+	−	+	14.886	14.810	14.868	14.876	14.958	14.932	14.888	0.003	-5.917
−	+	−	−	14.182	14.172	14.126	14.274	14.154	14.082	14.165	0.004	-5.485
−	+	+	+	14.758	14.784	15.054	15.058	14.938	14.936	14.921	0.016	-4.107
−	+	+	−	13.996	13.988	14.044	14.028	14.108	14.060	14.037	0.002	-6.237
+	−	−	+	15.272	14.656	14.258	14.718	15.198	15.490	14.932	0.215	-1.538
+	−	−	−	14.324	14.092	13.536	13.588	13.964	14.328	13.972	0.121	-2.116
+	−	+	+	13.918	14.044	14.926	14.962	14.504	14.136	14.415	0.206	-1.579
+	−	+	−	13.614	13.202	13.704	14.264	14.432	14.228	13.907	0.226	-1.487
+	+	−	+	14.648	14.350	14.682	15.034	15.384	15.170	14.878	0.147	-1.916
+	+	−	−	13.970	14.448	14.326	13.970	13.738	13.738	14.032	0.088	-2.430
+	+	+	+	14.184	14.402	15.544	15.424	15.036	14.470	14.843	0.327	-1.118
+	+	+	−	13.866	14.130	14.256	14.000	13.640	13.592	13.914	0.070	-2.653

Epi-layer Growth Experiment: Effect Estimates

Table 8: Factorial Effects, Original Epitaxial Layer Growth Experiment

Effect	$\bar{y}$	$\ln s^2$
A	-0.055	3.834
B	0.142	0.078
C	-0.109	0.077
D	0.836	0.632
AB	-0.032	-0.428
AC	-0.074	0.214
AD	-0.025	0.002
BC	0.047	0.331
BD	0.010	0.305
CD	-0.037	0.582
ABC	0.060	-0.335
ABD	0.067	0.086
ACD	-0.056	-0.494
BCD	0.098	0.314
$ABCD$	0.036	0.109

Epi-layer Growth Experiment: Half-Normal Plots

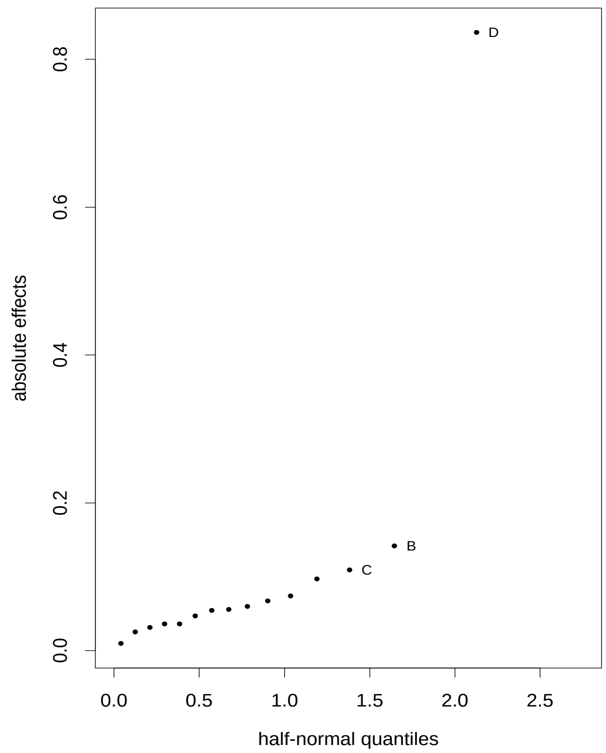

Figure 7 : Location effects

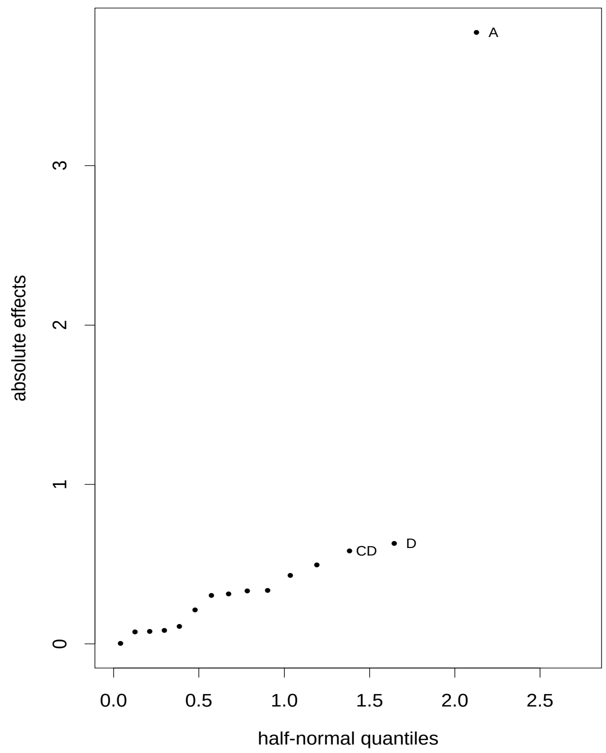

Figure 8 : Dispersion effects

Epi-layer Growth Experiment: Analysis and Optimization

- From the two plots, D is significant for $\bar{y}$ and A is significant for $z = \ln s^2$. D is an adjustment factor. Fitted models :

$$\begin{aligned}\hat{y} &= \hat{\alpha} + \hat{\beta}_D x_D = 14.389 + 0.418 x_D, \\ \hat{z} &= \hat{\gamma}_0 + \hat{\gamma}_A x_A = -3.772 + 1.917 x_A,\end{aligned}$$

- Two-step procedure:
  (i) Choose A at $-$ level (continuous rotation).
  (ii)Choose $x_D = 0.266$ to satisfy $14.5 = 14.389 + 0.418 x_D$
  If x_D = 30 and 40 seconds for $D = -$ and $+$, x_D = 0.266 would correspond to 35+0.266(5) = 36.33 seconds.

- Predicted variance

$$\hat{\sigma}^2 = \exp(-3.772 + 1.917(-1)) = (0.058)^2.$$

This is too optimistic! Predicted values should be validated with a *confirmation experiment*.

2^k Designs in 2^q Blocks

- Example: Arranging a 2^3 design in 2 blocks (of size 4). Use the 123 column in Table 9 to define the blocking scheme: block I if 123 = $-$ and block II if 123 = $+$. Therefore the block effect estimate $\bar{y}(II) - \bar{y}(I)$ is identical to the estimate of the 123 interaction $\bar{y}(123 = +) - \bar{y}(123 = -)$. The block effect B and the interaction 123 are called **confounded**. Notationally,

$$\mathbf{B} = \mathbf{123}.$$

- By giving up the ability to estimate 123, this blocking scheme increases the precision in the estimates of main effects and 2fi's by arranging 8 runs in two *homogeneous* blocks.

- Why sacrificing 123?
  ans: Effect hierarchy principle.

Arrangement of 2^3 Design in 2 Blocks

Table 9: Arranging a 2^3 Design in Two Blocks of Size Four

Run	**1**	**2**	**3**	**12**	**13**	**23**	**123**	Block
1	$-$	$-$	$-$	$+$	$+$	$+$	$-$	I
2	$-$	$-$	$+$	$+$	$-$	$-$	$+$	II
3	$-$	$+$	$-$	$-$	$+$	$-$	$+$	II
4	$-$	$+$	$+$	$-$	$-$	$+$	$-$	I
5	$+$	$-$	$-$	$-$	$-$	$+$	$+$	II
6	$+$	$-$	$+$	$-$	$+$	$-$	$-$	I
7	$+$	$+$	$-$	$+$	$-$	$-$	$-$	I
8	$+$	$+$	$+$	$+$	$+$	$+$	$+$	II

A 2^3 Design in 4 Blocks

- Similarly we can use $\mathbf{B_1 = 12}$ and $\mathbf{B_2 = 13}$ to define two independent blocking variables. The 4 blocks I, II, III and IV are defined by $B_1 = \pm$ and $B_2 = \pm$:

$\mathbf{B_2}$ \ $\mathbf{B_1}$	$-$	$+$
$-$	I	III
$+$	II	IV

- A 2^3 design in 4 blocks is given in Table 9. Confounding relationships: $B_1 = 12$, $B_2 = 13$, $B_1 B_2 = 12 \times 13 = 23$. Thus 12,13 and 23 are confounded with block effects and thus sacrificed.

Arranging a 2^3 Design in 4 Blocks

Table 10: Arranging a 2^3 Design in Four Blocks of Size Two

Run	**1**	**2**	**3**	$\mathbf{B_1}(=\mathbf{12})$	$\mathbf{B_2}(=\mathbf{13})$	**23**	**123**	block
1	$-$	$-$	$-$	$+$	$+$	$+$	$-$	IV
2	$-$	$-$	$+$	$+$	$-$	$-$	$+$	III
3	$-$	$+$	$-$	$-$	$+$	$-$	$+$	II
4	$-$	$+$	$+$	$-$	$-$	$+$	$-$	I
5	$+$	$-$	$-$	$-$	$-$	$+$	$+$	I
6	$+$	$-$	$+$	$-$	$+$	$-$	$-$	II
7	$+$	$+$	$-$	$+$	$-$	$-$	$-$	III
8	$+$	$+$	$+$	$+$	$+$	$+$	$+$	IV

Minimum Aberration Blocking Scheme

- On page 38, {I,12,13,23} forms the **block defining contrast subgroup** for the 2^3 design in 4 blocks. For a more complicated example (2^5 design in 8 blocks), see page 196 of WH.

- For any blocking scheme b, let $g_i(b)$ = number of i-factor interactions that are confounded with block effects. Must require $g_1(b) = 0$ (because no main effect should be confounded with block effects). For any two blocking schemes b_1 and b_2, let r = smallest i such that $g_i(b_1) \neq g_i(b_2)$. If $g_r(b_1) < g_r(b_2)$, b_1 is said to have *less aberration* than scheme b_2. (This is justified by the effect hierarchy principle). A blocking scheme has *minimum aberration* if no other blocking schemes have less aberration.

- Minimum aberration blocking schemes are given in Table 4A.1 of WH.

- Theory is developed under the assumption of no $block \times treatment$ interactions.

Comments on Board

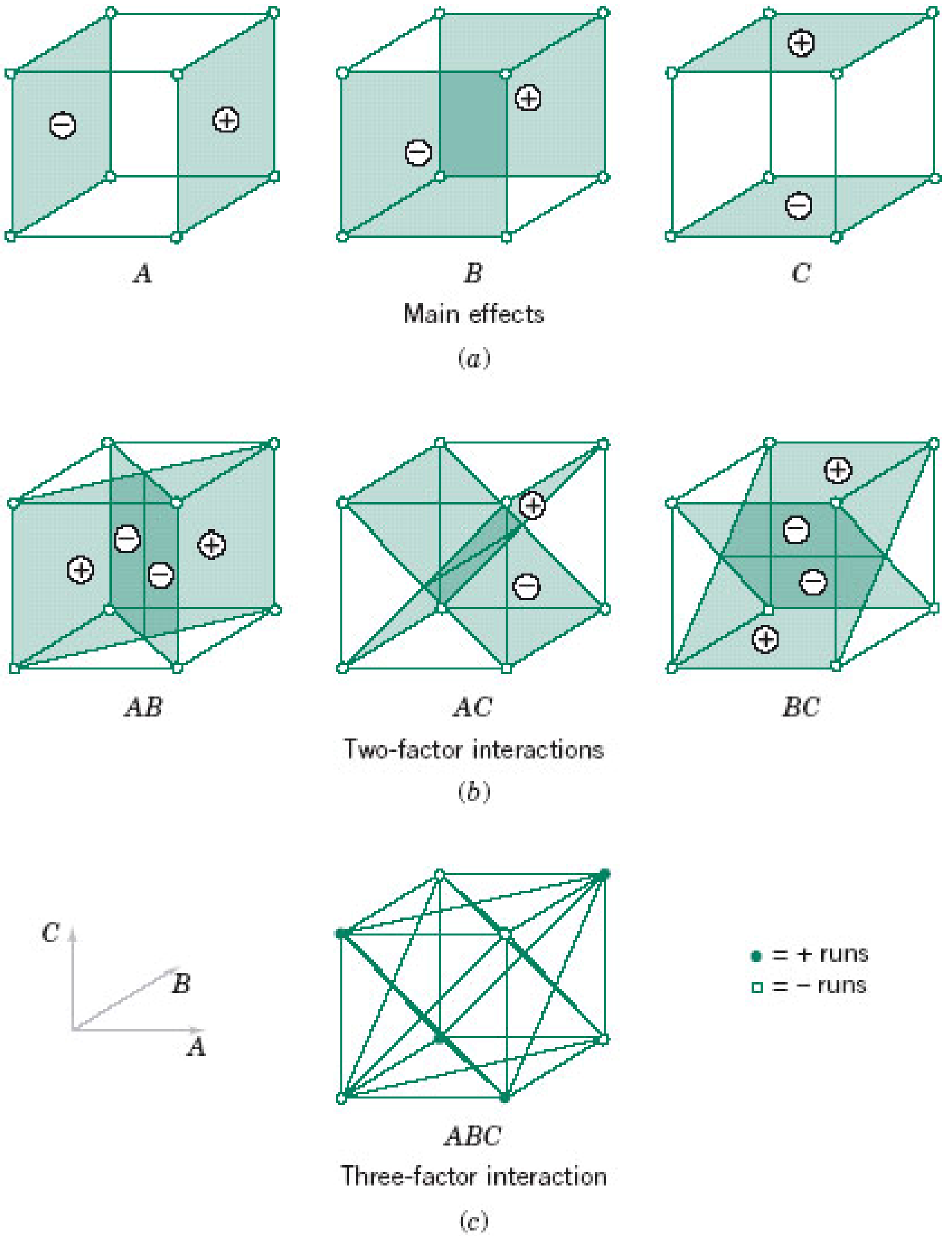

Figure 14-21 Geometric presentation of contrasts corresponding to the main effects and interaction in the 2^3 design. (a) Main effects. (b) Two-factor interactions. (c) Three-factor interaction.